B.Sc Part I (Hons) Paper I

Euler's Solution of a biquadratic equation

Let the biquadratic equation be

$$a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0$$

Which may be put in the form

$$z^4 + 6Hz^2 + 4Gz + a_0^2I - 3H^2 = 0 \quad \text{--- (1)}$$

Where $z = a_0x + a_1$

Let us assume that

$$z = \sqrt{p} + \sqrt{q} + \sqrt{r} \quad \text{--- (2)}$$

Squaring $z^2 = p+q+r + 2(\sqrt{p}\sqrt{q} + \sqrt{p}\sqrt{r} + \sqrt{q}\sqrt{r})$

i.e. $z^2 - (p+q+r) = 2(\sqrt{p}\sqrt{q} + \sqrt{q}\sqrt{r} + \sqrt{r}\sqrt{p})$

Squaring again, we get

$$\{z^2 - (p+q+r)\}^2 = 4(\sqrt{p}\sqrt{q} + \sqrt{q}\sqrt{r} + \sqrt{r}\sqrt{p})^2$$

$$\Rightarrow z^4 - 2z^2(p+q+r) + (p+q+r)^2$$

$$= 4[(pq+qr+rp) + 2(q\sqrt{p}\sqrt{r} + p\sqrt{q}\sqrt{r} + r\sqrt{p}\sqrt{q})]$$

$$= 4(pq+qr+rp) + 8\sqrt{p}\sqrt{q}\sqrt{r}(\sqrt{p}+\sqrt{q}+\sqrt{r})$$

$$= 4(pq+qr+rp) + 8\sqrt{p}\sqrt{q}\sqrt{r}\,z$$

$$= z^4 - 2z^2(p+q+r) - 8z\sqrt{p}\sqrt{q}\sqrt{r} + (p+q+r)^2 - 4(pq+qr+rp) = 0 \quad \text{--- (3)}$$

Comparing coefficients of like powers of z in (1) and (3), we get $p+q+r = -3H$,

$\sqrt{p}\sqrt{q}\sqrt{r} = -G/2$ i.e. $pqr = G^2/4$.

And $(\Sigma p)^2 - 4\Sigma pq = a_0^2 I - 3H^2$

i.e. $9H^2 - 4\Sigma pq = a_0^2I - 3H^2 \Rightarrow 4\Sigma pq = 12H^2 - a_0^2 I$

$$\therefore \Sigma pq = 3H^2 - \frac{a_0^2 I}{4}$$

Hence p, q, r are the roots of the cubic

$$t^3 + 3Ht^2 + \left(3H^2 - \frac{a_0^2 I}{4}\right)t - \frac{G^2}{4} = 0 \quad \text{--- (9)}$$

Which is called the Euler's cubic

Relation between the roots of the biquadratic and Euler's cubic.

We have shown in the previous article that we get only four values of $z = \sqrt{p} + \sqrt{q} + \sqrt{r}$ due to the relation $\sqrt{p}\sqrt{q}\sqrt{r} = -G/2$.

If G be $-$ve, $\sqrt{p}\sqrt{q}\sqrt{r}$ is positive and consequently in this case, either all of $\sqrt{p}, \sqrt{q}, \sqrt{r}$ should be +ve or two $-$ve and one +ve

Hence in this case, we get

$$\left.\begin{aligned} z_1 &= a_0\alpha + a_1 = \sqrt{p} - \sqrt{q} - \sqrt{r} \\ z_2 &= a_0\beta + a_1 = -\sqrt{p} + \sqrt{q} - \sqrt{r} \\ z_3 &= a_0\gamma + a_1 = -\sqrt{p} - \sqrt{q} + \sqrt{r} \\ z_4 &= a_0\delta + a_1 = \sqrt{p} + \sqrt{q} + \sqrt{r} \end{aligned}\right\} \text{--- (A)}$$

From (A)

$$a_0(\beta + \gamma) + 2a_1 = -2\sqrt{p}, \quad a_0(\alpha + \delta) + 2a_1 = 2\sqrt{p}$$

$$\therefore a_0(\beta + \gamma - \alpha - \delta) = -4\sqrt{p} \Rightarrow a_0^2(\beta + \gamma - \alpha - \delta)^2 = 16p$$

$$\left.\begin{aligned} \therefore p &= \frac{a_0^2}{16}(\beta + \gamma - \alpha - \delta)^2 \\ \text{Similarly } q &= \frac{a_0^2}{16}(\gamma + \alpha - \beta - \delta)^2 \\ \text{and } r &= \frac{a_0^2}{16}(\alpha + \beta - \gamma - \delta)^2 \end{aligned}\right\} \text{--- (B)}$$

The equation (B) gives the necessary relations between the roots of the biquadratic and those of the Euler's cubic.

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